ELSEVIER
Discrete Applied Mathematics 64 (1996) 267–279
DISCRETE APPLIED MATHEMATICS

Design of a d-connected digraph with a minimum number of edges and a quasiminimal diameter: II

Terunao Soneoka*, Makoto Imase, Yoshifumi Manabe

NTT Software Laboratories, 3-9-11 Midori-Cho, Musashino-Shi, Tokyo, 180 Japan

Received 1 February 1991; revised 20 September 1994

Abstract

For designing reliable and efficient communications networks, the problem of constructing a maximally connected d-regular digraph (directed graph) with a small diameter is investigated. A maximally connected d-regular digraph with a diameter at most two larger than the lower bound for any number of nodes $n \geqslant 2d$ and any $d \geqslant 3$ is constructed. Since the diameter of this digraph is *quasiminimal* (at most one larger than the lower bound) for $n \leqslant d^3 + d$, we can construct maximally connected d-regular digraphs with a quasiminimal diameter for any n $(> d)$ and d, even for those cases not covered in previous papers.

1. Introduction

Two fundamental concerns in designing a communications network or a multiprocessor network are the overall reliability and the maximum transmission delay. These can be respectively measured by the connectivity and the diameter of a graph representing the network [2, 9, 12, 13].

Several studies have treated the problem of constructing a maximally connected d-regular graph or digraph with a small diameter [11–14]. Nearly optimal solutions attaining a diameter of approximately twice the lower bound have been presented by Schumacher for undirected graphs [12] and by Sengupta et al. for directed graphs [13].

This paper studies the problem of constructing a maximally connected d-regular *digraph* with a *quasiminimal* (at most one larger than the lower bound) diameter for any n $(> d)$ and d. The justification of such a problem for digraphs can be found in our previous paper [14]. For d-regular digraphs G with n nodes, the lower bound of the diameter $D_L(n, d)$ is given as

$$D_L(n, d) = \lceil \log_d(n(d-1)+d) \rceil - 1, \tag{1}$$

* Corresponding author. Tel.: + 81-422-59-2578; fax.: + 81-422-593409.

SSDI 0166-218X(94)00113-8

where $1 < d < n-1$ and $\lceil x \rceil$ denotes the minimum integer not less than x [8]. This equation can be easily derived from the *Moore bound* for digraphs and the non-existence of *Moore digraphs* other than for $d = 1$ and $d = n-1$ [3].

Reddy et al. [1] proposed a method for constructing a maximally connected 2-regular digraph D_n with a quasiminimal diameter for any n by modifying the generalized de Bruijn digraph $G_{\mathrm{B}}(n, d)$ [5, 7, 9, 11] with degree $d = 2$. Furthermore, Soneoka et al. [14] proposed a maximally connected d-regular digraph with n nodes and a quasiminimal diameter for $d \geqslant 3$ and any $n > d^3$ by modifying $G_{\mathrm{B}}(n, d)$. Du et al. [4], on the other hand, independently proposed a similar modification of consecutive-d digraphs, a more general class of digraphs including the generalized de Bruijn digraph and the generalized Kautz digraph [8]. (A similar general class of digraphs, c-circulant digraphs, can be found in [10].) By their method, a maximally connected d-regular digraph with a quasiminimal diameter can be constructed for $d \geqslant 2$ if n is divided by d and $n \geqslant d^2$.

For any $n \leqslant d^3$, however, we cannot obtain a unified method of constructing maximally connected d-regular digraphs only by replacing all self-loops in $G_{\mathrm{B}}(n, d)$ by a cycle. This is because the connectivity of some G_{B}, for example $G_{\mathrm{B}}(d^2 - d, d)$, is less than $d-1$. To construct maximally connected d-regular digraphs for any $n \leqslant d^3$, another method is required.

In this paper, such a digraph $G_{\mathrm{S}}(n, d)$ with a diameter at most two larger than the lower bound for any $n \geqslant 2d$ and any $d \geqslant 3$ is constructed by using an algorithm inspired by the algorithm of Sengupta et al. [13]. Let n be $md + t$ $(0 \leqslant t < d)$. First, a maximally edge-connected d-regular digraph $G^*_{\mathrm{B}}(m, d)$ with m $(\geqslant 2)$ nodes and a quasiminimal diameter is constructed. Then, construct the line digraph of $G^*_{\mathrm{B}}(m, d)$, $L(G^*_{\mathrm{B}})$, with md nodes, and connect t additional nodes to it. This yields the proposed digraph $G_{\mathrm{S}}(md + t, d)$ with a diameter not larger than $\lceil \log_d m \rceil + \lceil t/d \rceil + 1$, which is at most two larger than the lower bound $D_{\mathrm{L}}(n, d)$. The diameter of this digraph $D(G_{\mathrm{S}}(n, d))$ is quasiminimal for $n \leqslant d^3 + d$ because $D(G_{\mathrm{S}}(n, d)) \leqslant 3$ and $D_{\mathrm{L}}(n, d) = 2$ for $2d \leqslant n \leqslant d^2 + d$, and $D(G_{\mathrm{S}}(n, d)) \leqslant 4$ and $D_{\mathrm{L}}(n, d) = 3$ for $d^2 + d < n \leqslant d^3 + d$. We can therefore cover the cases which could not be covered by previous papers [4, 8, 11, 14]. Note that for $n \leqslant 2d + 1$, a maximally connected d-regular digraph with n nodes and a quasiminimal diameter can be easily constructed (for example, the digraph proposed in [1]).

The following section defines several digraph terms used in this paper. Section 3 presents a method for constructing a maximally edge-connected d-regular digraph $G^*_{\mathrm{B}}(m, d)$ with a quasiminimal diameter. Section 4 presents a maximally connected d-regular digraph $G_{\mathrm{s}}(n,d)$ with a diameter at most two larger than the lower bound.

2. Definitions

Let $G = (V, E)$ be a *directed pseudograph* where V is a set of nodes and E is a set of (directed) edges, which may contain a *self-loop*, an edge from a node to itself, and *multiple edges*, that is several edges from one node to another. A directed pseudograph not containing self-loops is called a *directed multigraph*. Hereafter, a directed pseudograph will be called a digraph for short. For a node v, its out-degree (in-degree) is the number of edges which are incident out of (into) node v without counting self-loops. A digraph G is called *d-regular* if the out- and in-degrees of every node are equal to d.

A *walk* p in G is an alternating sequence of nodes and edges, $v_0, e_0, v_1, \ldots, v_{i-1}, e_i, v_i, \ldots, e_k, v_k$, where $e_i = (v_{i-1}, v_i)$. v_0 and v_k are respectively called the *initial* and the *terminal* node of p. The *distance* from a node u to a node v, denoted by $dis(u, v)$, is the length of (the number of edges in) a shortest walk from u to v. For a node-subset $W \subset V$ and a node v in $V - W$, $dis(v, W)$ denotes $\min\{dis(v, u) \mid u \in W\}$, while $dis(W, v)$ denotes $\min\{dis(u, v) \mid u \in W\}$. The *diameter* of G, $D(G)$, is the maximum distance between any pair of nodes.

A digraph G is said to be *strongly connected* if there is a walk between every pair of distinct nodes. The *connectivity* $\kappa(G)$ (*edge-connectivity* $\lambda(G)$) of the digraph G is defined as the minimum number of nodes (resp. edges) whose removal results in a trivial or not strongly connected digraph. For a d-regular digraph G,

$$\kappa(G) \leqslant \lambda(G) \leqslant d.$$

A d-regular digraph G is called to be *maximally connected* if $\kappa(G) = d$ and *maximally edge-connected* if $\lambda(G) = d$.

If (u, v) is an edge, then u is called a *predecessor* of v; similarly, v is called a *successor* of u. For a node subset $V' \subseteq V$, $S(V')$ is defined as the set of successors of V', and $P(V')$ is defined as the set of predecessors of V'. Further, $S^t(V')$ is defined as $S(S^{t-1}(V'))$ for $t \geqslant 1$ and $S^0(V') = V'$, while $P^t(V')$ is defined as $P(P^{t-1}(V'))$ for $t \geqslant 1$ and $P^0(V') = V'$. In other words, $S^t(V')$ is the set of nodes to which there is a t-length walk from some node v in V', while $P^t(V')$ is the set of nodes from which there is a t-length walk to some node v in V'.

A *matching* is defined as a set of edges such that no two edges of it are adjacent. A digraph $G = (V, E)$ is said to be a *bipartite* digraph $G = (X, Y, E)$ if $V = X \cup Y$, $X \cap Y = \emptyset$, and each edge in E joins a node of X and a node of Y.

3. Construction method of $G_B^*(m, d)$

This section presents the construction method of a maximally edge-connected d-regular digraph $G_B^*(m, d)$ with a quasiminimal diameter.

Construction method of $G_B^*(m, d)$

Let $G_B(m, d) = (V, E)$ ($m \geqslant 2, d \geqslant 2$) be the generalized de Bruijn digraph, where $V = \{0, 1, \ldots, m-1\}$ and $E = \{(u, v) \mid v \equiv ud + a (\mathrm{mod}\, m),\ a = 0, 1, \ldots, d-1\}$.

For any $v \in V$, the number of self-loops of v is $\lfloor d/m \rfloor$ or $\lceil d/m \rceil$, and at least two nodes (0 and $m-1$) have $\lceil d/m \rceil$ self-loops. $G_B^*(m, d)$ is a directed multigraph constructed from $G_B(m, d)$ by removing all the self-loops and adding $\lfloor d/m \rfloor$ cycles, each

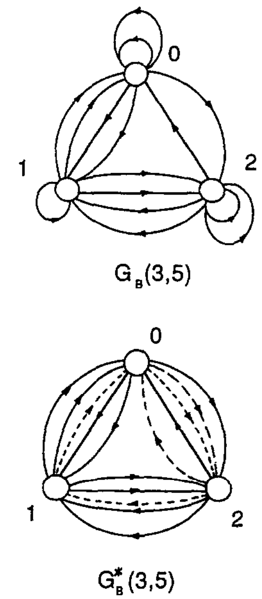


Fig. 1. $G_B(3,5)$ and $G_B^*(3,5)$.

connecting every node, and another cycle that will connect the nodes originally with $\lfloor d/m \rfloor + 1$ self-loops.

A directed multigraph $G_{\mathrm{B}}^*(3, 5)$ constructed from $G_{\mathrm{B}}(3, 5)$ is shown in Fig. 1 and a digraph $G_{\mathrm{B}}^*(12, 3)$ constructed from $G_{\mathrm{B}}^*(12, 3)$ is shown in Fig. 2.

The following theorem will be proved.

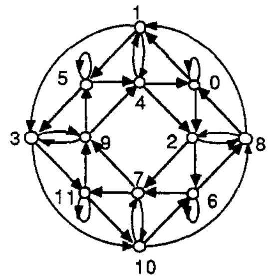
Theorem 1. *$G_{\mathrm{B}}^*(m, d)$ is a maximally edge-connected d-regular digraph with a diameter $D(G_{\mathrm{B}}^*)$ not larger than $\lceil \log_d m \rceil$ for $m \geqslant 2$ and $d \geqslant 3$. (Namely, $d^{D(G_{\mathrm{B}}^*)-1} < m$.)*

The following properties of $G_{\mathrm{B}}(m, d)$ will be needed in the proof of Theorem 1.

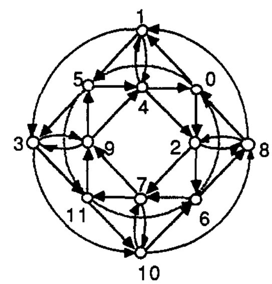
Property 1 (Imase et al. [9]). *Let v be a node in $G_{\mathrm{B}}(m, d)$ and $D(G_{\mathrm{B}})$ be the diameter of G_{B}. If $V' \subseteq S^{t-1}(v)$ and $1 \leqslant t < D(G_{\mathrm{B}})$, then $|S(V')| = d \cdot |V'|$, and if $V' \subseteq P^{t-1}(v)$ and $1 \leqslant t < D(G_{\mathrm{B}})$, then $|P(V')| = d \cdot |V'|$.*

Property 2 (Reddy et al. [11]). *The number of self-loops in $G_{\mathrm{B}}(m, d)$ is $d + \gcd(m, d-1) - 1$, where $\gcd(p, q)$ is the greatest common divisor for p and q.*

Proof of Theorem 1. It is clearly valid that G_{B}^* is d-regular. Since the previous paper [7] showed that $D(G_{\mathrm{B}}(m, d)) = \lceil \log_d m \rceil$, it is clear that $D(G_{\mathrm{B}}^*(n, d)) \leqslant \lceil \log_d m \rceil$.



$G_B(12,3)$



$G_B^*(12,3)$

Fig. 2. $G_B(12,3)$ and $G_B^*(12,3)$.

Let $d = mp + q$ ($p \geqslant 0$ and $0 \leqslant q < m$), $G_B^*(m, d) = (V, E)$, $G_B^*(m, m) = (V, E_1)$, and $G_B^*(m, q) = (V, E_2)$, then E contains p multiple edges from u to v, where u and v are the nodes such that $(u, v) \in E_1$, and another edge from u' to v', where u' and v' are the nodes such that $(u', v') \in E_2$. Thus,

$$d \geqslant \lambda(G_B^*(m, d)) \geqslant p \cdot \lambda(G_B^*(m, m)) + \lambda(G_B^*(m, q)).$$

Since $G_B^*(m, m)$ has an edge between any pair of nodes, and $G_B^*(m, m)$ is obtained from $G_B^*(m, m)$ by removing self-loops and adding a cycle connecting every node, it is valid that $\lambda(G_B^*(m, m)) = m$. Hence, to prove $\lambda(G_B^*(m, d)) = d$, it is enough to show that $\lambda(G_B^*(m, q)) = q$.

The following will prove $\lambda(G_B^*(m, d)) = d$ for $d < m$. Let G_B^{**} be a digraph obtained from $G_B(m, d)$ by returning all the removed self-loops (retaining the added cycle as well). To prove $\lambda(G_B^*(m, d)) = d$, it is enough to prove $\lambda(G_B^{**}) = d$, because self-loops do not contribute to the value of edge-connectivity. Remark that G_B^{**} also remains d-regular. For $G_B^{**} = (V, E)$, let $T \subset E$ be an arbitrary edge-cut of G_B^{**}. The node set V can be partitioned into two disjoint non-empty sets Y and Y' such that $G_B^{**} - T$ contains no edges from Y to Y' and every edge of T has initial node in Y and terminal node in Y'. Let Y_0 (Y_0') be the set of the initial (terminal) nodes of the edges of T.

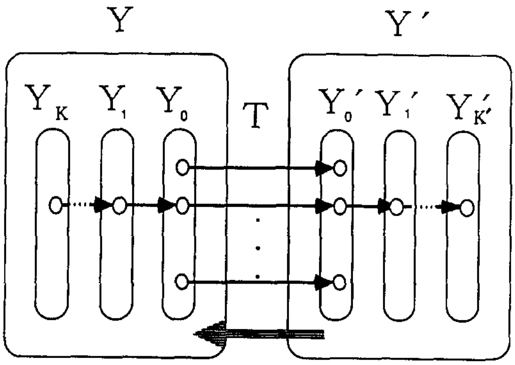


Fig. 3. Structure of digraph with edge cut T.

Let $K = \max_{y \in Y} dis(y, Y_0)$, $K' = \max_{y' \in Y'} dis(Y'_0, y')$, $Y_i = \{y \in Y \mid dis(y, Y_0) = i\}$ $(1 \leqslant i \leqslant K)$, and $Y'_i = \{y' \in Y' \mid dis(Y'_0, y') = i\}$ $(1 \leqslant i \leqslant K')$. Thus $|Y_0| \leqslant |T|$, $|Y'_0| \leqslant |T|$, $|Y_{i+1}| \leqslant d|Y_i|$ $(0 \leqslant i \leqslant K-1)$, and $|Y'_{i+1}| \leqslant d|Y'_i|$ $(0 \leqslant i \leqslant K'-1)$ (see Fig. 3).

Denote $D(G_B^{**})$ as D, and remark that $K + K' + 1 \leqslant D$. Since G_B^{**} is d-regular, the number of edges from Y to Y' is equal to that from Y' to Y. Thus, $K \leqslant K'$ can be supposed without loss of generality.

Case 1: $K = 0$.

This case indicates $Y = Y_0$, thus $1 \leqslant |Y| \leqslant |T|$. Let $T' = T \cap E'$ and $T'' = T \cap E''$, where E' is the set of edges originally in G_B and E'' is the set of edges in the added cycle, then $T = T' \cup T''$. Let the subgraph of G_B^{**}, (V, E'), be denoted by G'_B. For $y \in Y$, let $E'(y) = \{(y, y') \mid y' \in Y' \text{ and } (y, y') \in E'\}$, and $deg'_+(y)$ be the out-degree of $y \in Y$ in G'_B. Since $G_B(m, d)$ has no multiple edge when $m > d$,

$$|T| = |T'| + |T''| = \sum_{y \in Y} |E'(y)| + |T''| \geqslant \sum_{y \in Y} (deg'_+(y) - (|Y| - 1)) + |T''|.$$

When $|T''| = 0$, Y contains every node with a self-loop or no node with a self-loop. Since there is not a node with two or more self-loops when $d < m$, from Property 2, the number of nodes with a self-loop is not less than d. If Y contains every node with a self-loop, $|Y| \geqslant d$. From $|T| \geqslant |Y|$, this results in $|T| \geqslant d$. If Y contains no node with a self-loop, since $deg'_+(y) = d$ for any node y not having a self-loop, and $|T''| = 0$,

$$|T| \geqslant \sum_{y \in Y} (deg'_+(y) - (|Y| - 1)) = |Y|(d - |Y| + 1).$$

From $1 \leqslant |Y| \leqslant |T|$, we get $|T| \geqslant d$.

When $|T''| \geqslant 1$, since $deg'_+(y) \geqslant d - 1$,

$$|T| \geqslant \sum_{y \in Y} (deg'_+(y) - (|Y| - 1)) + |T''| \geqslant |Y|(d - |Y|) + 1.$$

From $1 \leqslant |Y| \leqslant |T|$, we get $|T| \geqslant d$.

Case 2: $K \geqslant 1$ (*and therefore* $D \geqslant 3$).

Let y be a node of Y such that $dis(y, Y_0) = K$, i.e., $y \in Y_k$. Since $S(V') \subseteq Y \cup Y_0'$ for any $V' \subseteq Y$, $|Y_0'| \leqslant |T|$, and if $y \in Y_k$ ($k \geqslant 1$), then $S^k(y) = S^k(y) \cap Y$,

$$\begin{aligned}|S^{K+1}(y) \cap Y| &= |S(S^K(y)) \cap Y| = |S(S^K(y) \cap Y) \cap Y| \\ &= |S(S^K(y) \cap Y)| - |S(S^K(y) \cap Y) \cap Y_0'| \\ &\geqslant |S(S^K(y) \cap Y)| - |Y_0'| \geqslant |S(S^K(y) \cap Y)| - |T|.\end{aligned}$$

Since $|S^K(y) \cap Y| = d^K$ in the subdigraph G_B' of G_B^{**} from Property 1, it is valid that $|S^K(y) \cap Y| \geqslant d^K$ in G_B^{**}. From this and Property 1,

$$|S^{K+1}(y) \cap Y| \geqslant |S(S^K(y) \cap Y)| - |T| \geqslant d|S^K(y) \cap Y| - |T| \geqslant d^{K+1} - |T|.$$

Similarly, for $K + 1 \leqslant t \leqslant D - 1$,

$$\begin{aligned}|S^t(y) \cap Y| &= |S(S^{t-1}(y)) \cap Y| \geqslant |S(S^{t-1}(y) \cap Y) \cap Y| \\ &\geqslant |S(S^{t-1}(y) \cap Y)| - |T| \geqslant d|S^{t-1}(y) \cap Y| - |T|.\end{aligned}$$

Thus

$$|Y| \geqslant |S^{D-1}(y) \cap Y| \geqslant d^{D-1} - |T| \frac{\mathrm{d}^{D-1-K} - 1}{d-1}. \tag{2}$$

On the other hand,

$$|Y| = |Y_0| + \sum_{i=1}^{K} |Y_i| \leqslant |T| + |T| \sum_{i=1}^{K} d^i = |T| \frac{\mathrm{d}^{K+1} - 1}{d-1}. \tag{3}$$

From (2) and (3),

$$|T| \frac{d^{K+1} - 1}{d-1} \geqslant d^{D-1} - |T| \frac{d^{D-1-K} - 1}{d-1}.$$

Thus,

$$|T| \geqslant \frac{(d-1)\, d^{D-1}}{d^{K+1} + d^{D-1-K} - 2}. \tag{4}$$

In the case of $K \geqslant 2$, from (4), $D \geqslant K + K' + 1 \geqslant 2K + 1$, and $d \geqslant 3$,

$$|T| \geqslant \frac{(d-1)\, d^{2K}}{d^{K+1} + d^K - 2} \geqslant \frac{(d-1)\, d^4}{d^3 + d^2 - 2} = \frac{d^4}{d^2 + 2d + 2} > d.$$

In the case of $K = 1$ and $D \geqslant 4$, from (4) and $d \geqslant 3$,

$$|T| \geqslant \frac{(d-1)\, d^3}{2d^2 - 2} = \frac{d^3}{2(d+1)} > d.$$

Hence, the proof of this theorem will be completed if the case of $K = 1$ and $D = 3$ can be shown. In this case, from $K + K' + 1 \leqslant D$, $K' = 1$. From (2).

$$|Y| \geqslant d^2 - |T|. \tag{5}$$

Similar to the derivation of (5),

$$|Y'| \geqslant d^2 - |T|. \tag{6}$$

On the other hand, let $a = \min_{y \in Y_1} |T(y)|$, where $T(y) = \{y'_0 \in Y'_0 \mid dis(y, y'_0) = 2\}$. Thus $1 \leqslant a \leqslant |Y'_0| \leqslant |T|$. Let $(\alpha_1, \ldots, \alpha_p)$ $(p = |Y_0|)$ be the degrees toward Y'_0 of the nodes of Y_0, and let the number of paths of length 2 from any node of Y_1 to any node of Y'_0 be l. Since $\sum_{i=1}^{p} \alpha_i = |T|$, we get $l \leqslant \sum_{i=1}^{p} \alpha_i \cdot d \leqslant |T| d$. By the definition of a, $|Y_1| \leqslant l/a \leqslant |T| d/a$. From this and $|Y_0| \leqslant |T|$,

$$|Y| = |Y_0| + |Y_1| \leqslant |T| + |T| \frac{d}{a}. \tag{7}$$

Let y be a node in Y_1 such that $|T(y)| = a$. Since $dis(y, y') \leqslant D = 3$ for any $y' \in Y'_1$, we get $|Y'_1| \leqslant a \cdot d$. From this and $|Y'_0| \leqslant |T|$,

$$|Y'| = |Y'_0| + |Y'_1| \leqslant |T| + a \cdot d. \tag{8}$$

From (5)–(8),

$$d^2 - |T| \leqslant |T| + a \cdot d \quad \text{if } 1 \leqslant a \leqslant \sqrt{|T|},$$

$$d^2 - |T| \leqslant |T| + |T| \frac{d}{a} \quad \text{if } \sqrt{|T|} \leqslant a \leqslant |T|.$$

Thus, $|T| \geqslant d^2/4$. From $d \geqslant 3$, it is valid that $|T| \geqslant d$. □

That the diameter of $G^*_{\mathrm{B}}(m, d)$ is quasiminimal can be seen from (1).

4. Construction method of $G_{\mathrm{S}}(n, d)$

This section proposes a construction method of a maximally connected d-regular digraph $G_{\mathrm{S}}(n, d)$ with $n = md + t$ $(0 \leqslant t < d)$ nodes and a diameter not larger than $\lceil \log_d m \rceil + \lceil t/d \rceil + 1$. From (1), $D(G_{\mathrm{S}}(n, d))$ is at most two larger than the lower bound, and especially for $n \leqslant d^3 + d$, $D(G_{\mathrm{S}}(n, d))$ is quasiminimal.

First, the line digraph of $G^*_{\mathrm{B}}(m, d)$, $L(G^*_{\mathrm{B}}(m, d))$, is constructed, which is a d-regular digraph with md nodes. Next, by adding t $(< d)$ nodes, we produce $G_{\mathrm{S}}(md + t, d)$. In the line digraph $L(G)$ of a digraph $G = (V, E)$, each node represents an edge of G, that is $V(L(G)) = \{uv \mid (u, v) \in E(G)\}$, and the node uv is adjacent to the node wz iff $v = w$ (i.e., when the edge (u, v) is adjacent to the edge (w, z) in G) [6]. Fig. 4 shows a digraph G and its line digraph $L(G)$.

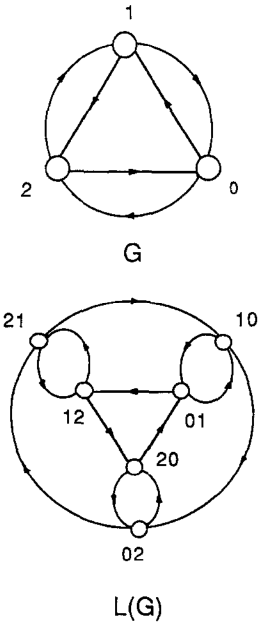


Fig. 4. A digraph and its line digraph.

Construction method of $G_S(md + t, d)$

(i) $t = 0$:

$$G_S(md, d) = L(G_B^*(m, d)).$$

(ii) $1 \leqslant t < d$:

Choose an arbitrary node v of $G_B^*(m, d)$. In $G_S(md, d) = (V', E')$, let X be the set of nodes corresponding to the in-edges of v, and Y be that corresponding to the out-edges of v. Since G_B^* is d-regular, $|X| = |Y| = d$, let $X = \{x_0, x_1, \ldots, x_{d-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{d-1}\}$. From the construction method of the line digraph, E' contains an edge from any node in X to any node in Y, let $M = \{(x, y) \mid \forall x \in X, \forall y \in Y\}$.

$G_S(md + t, d) = (V'', E'')$ is constructed by adding t nodes $W = \{w_0, w_1, \ldots, w_{t-1}\}$ to $G_S(md, d) = (V', E')$ as follows:

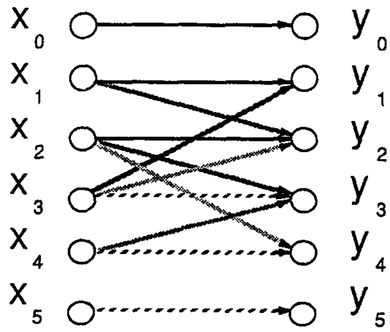
$$V'' = V' \cup W,$$

$$E'' = E' \cup \{(w_i, w_j) \mid i \neq j\} \cup \bigcup_{i=0}^{t-1} \{(x, w_i), (w_i, y) \mid x \in X_i, y \in Y_i\} - \bigcup_{i=0}^{t-1} M_i,$$

where

$$M_i = \{(x_{i+p}, y_{i+q}) \mid q \equiv i + p \pmod{d - t + 1},\ p = 0, 1, \ldots, d - t,\ 0 \leqslant q \leqslant d - t\}, \tag{9}$$

$$X_i = \{x_i, x_{i+1}, \ldots, x_{i+d-t}\} \quad \text{and} \quad Y_i = \{y_i, y_{i+1}, \ldots, y_{i+d-t}\},$$



$M_0 = \{(x_0, y_0), (x_1, y_1), (x_2, y_2)\}$

$M_1 = \{(x_1, y_2), (x_2, y_3), (x_3, y_1)\}$

$M_2 = \{(x_2, y_4), (x_3, y_2), (x_4, y_3)\}$

$M_3 = \{(x_3, y_3), (x_4, y_4), (x_5, y_5)\}$

Fig. 5. M_i $(i = 0, 1, \ldots, t-1)$ for $d = 6$ and $t = 4$.

for $i = 0, 1, \ldots, t-1$ $(t < d)$. (Note that X_i and Y_i are, respectively, the initial and the terminal node sets of M_i.)

Fig. 5 shows M_i $(i = 0, 1, \ldots, t-1)$ for $d = 6$ and $t = 4$.

Before proving the diameter and the connectivity of $G_S(md, +t, d)$, the properties of the line digraph $L(G)$ and of M_i will be prepared.

Property 3 (Fiol et al. [6]). *Given a maximally edge-connected d-regular* $(d > 1)$ *digraph G with m nodes, its line digraph L(G) is a d-regular digraph with*

(a) $m \cdot d$ *nodes,*

(b) $D(L(G)) = D(G) + 1$, *and*

(c) $\kappa(L(G)) = \lambda(G) = d$.

Property 4. (a) *M_i is a matching of the bipartite digraph $H = (X, Y, M)$.*

(b) $M_i \cap M_j = \emptyset$ $(i \neq j)$.

(c) $|\bigcup_{i \in I} X_i| = |\bigcup_{i \in I} Y_i| \geqslant d - t + |I|$ *for any* $I \subseteq \{0, 1, \ldots, t-1\}$.

Proof. (a) is clearly valid. We will prove $M_i \cap M_j = \emptyset$ $(i \neq j)$. Assume that $(x_{i+p}, y_{i+q}) \in M_i$ is coincident with $(x_{j+r}, y_{j+s}) \in M_j$, that is $i + p = j + r$ and $i + q = j + s$. These equalities, $q \equiv i + p \pmod{d - t + 1}$, and $s \equiv j + r \pmod{d - t + 1}$ contradict $i \neq j$. Hence, (b) is valid. Next, we will prove (c). From the definition, $|X_i| = |Y_i| = d - t + 1$. For any $I \subseteq \{0, 1, \ldots, t-1\}$, let I' be $I - \{j\}$ where j is a maximum element of I. Since the element of X_j with a maximum suffix, x_{j+d-t}, is not contained in $\bigcup_{i \in I'} X_i$, $|(\bigcup_{i \in I'} X_i) \cup X_j| \geqslant |\bigcup_{i \in I'} X_i| + 1$. Similarly, $|(\bigcup_{i \in I'} Y_i) \cup Y_j| \geqslant |\bigcup_{i \in I'} Y_i| + 1$. Thus, we can derive the inequality of (c). □

We now show the following theorem:

Theorem 2. *$G_S(md + t, d)$ $(0 \leqslant t < d)$ is a maximally connected d-regular digraph with a diameter $D(G_S(md + t, d))$ not larger than $\lceil \log_d m \rceil + \lceil t/d \rceil + 1$.*

Proof. From the construction of $G_S(md + t, d)$ and Property 4(b), it is clear that $G_S(md + t, d)$ is d-regular. When $t = 0$, from Property 3 and Theorem 1, we get $D(G_S(md, d)) = D(G_B^*(m, d)) + 1 \leqslant \lceil \log_d m \rceil + 1$ and $\kappa(G_S(md, d)) = \lambda(G_B^*(m, d)) = d$.

Next we will consider the case $t > 0$. First, since $D(G_S(md, d)) \leqslant \lceil \log_d m \rceil + 1$, we will show that $D(G_S(md + t, d)) \leqslant D(G_S(md, d)) + 1$, and next $\kappa(G_S(md + t, d)) = d$. Denote $G_S(md, d)$ by $G'_S = (V', E')$, $G_S(md + t, d)$ by $G''_S = (V'', E'')$, and $\bigcup_{i=0}^{t-1} M_i$ by M^*. Note that $V'' = V' \cup W$.

(1) First we prove that $D(G''_S) \leqslant D(G'_S) + 1$. Consider any two nodes u and v in V'. Let p be the shortest walk from u to v in G'_S. When p does not contain an edge of M^*, the walk length in G''_S remains equal to the length of p. When p contains only one edge of M^*, the walk length in G''_S increases from the length of p by 1. That more than one edge of M^* are contained in p is impossible since p is a shortest walk and G'_S is a line digraph. Hence, the distance between any two nodes in V' in G''_S is not larger than $D(G'_S) + 1$.

The distance between any w_i and w_j is 1. The length of the walk from any $u \in V'$ to any $w_i \in W$ will be considered. Let v be a predecessor of w_i in V' and p be the shortest walk from u to v in G''_S. If p is the same as that in G'_S, then there is a walk from u to w_i not larger than $D(G'_S) + 1$. Otherwise, the walk p goes through some w_j. Hence, there is a walk $u, \ldots, w_j, w_i$, whose length is not larger than $D(G'_S)$. Similar is the case from any w_i to any $u \in V'$ by considering a successor of w_i. Consequently, $D(G''_S) \leqslant D(G'_S) + 1$.

(2) Next we prove that $\kappa(G''_S) = d$. To prove $\kappa(G''_S) = d$, it will be shown that even if any $d - 1$ nodes are removed from G''_S, there exists a walk from any remaining node u to any remaining node v. Let F be the removed $d - 1$ nodes. Denote $F \cap V'$ by F_V and $F \cap W$ by F_W, and let $|F_V|$ be s and $|F_W|$ be k. Then $F = F_V \cup F_W$ and $s + k = d - 1$.

First, the case of $u \in V'$ and $v \in V' - Y$ will be considered. Since $\kappa(G'_S) = d$, there are d disjoint walks from u to v in G'_S. From this, $|F_V| = s$, and $d - s = k + 1$, there exist at least $k + 1$ disjoint walks $p_1, \ldots, p_{k+1}, \ldots$ from u to v in $G'_S - F_V$. When there is a p_i which does not go through any edge of M^*, p_i remains a walk from u to v in $G''_S - F_V - F_W$.

In the other case, every p_i goes through some edge of M^*. Let P be the set of first nodes of X which occurred in these disjoint walks $p_1, \ldots, p_{k+1}, \ldots$, and Q be the set of last nodes of Y which occurred in $p_1, \ldots, p_{k+1}, \ldots$. If G'_S contains an edge from some $p \in P$ to some $q \in Q$ which is not contained in M^*, there is a walk $u, \ldots, p, q, \ldots, v$ in $G_S'' - F_V - F_W$. Otherwise, let p be a node in P, G_S' contains edges from p to every $q \in Q$, which are contained in M^*. Since the out-edges of p are not contained in the same M_i in G_S' from Property 4(a), there are walks from p to every $q \in Q$ which go through distinct nodes of W in G_S''. Since $v \notin Y$, $|Q| = |\{p_i\}| \geqslant k+1$. Thus, the number of these walks in G_S'' is at least $k+1$. From $|F_W| = k$, there is a walk from p to some $q \in Q$ via some $w_i \in W - F_W$ in $G_S'' - F_V - F_W$. Consequently, there is a walk $u, \ldots, p, w_i, q, \ldots, v$ in $G_S'' - F$.

Next, the case of $v \in Y$ will be considered. In this case, the set of predecessors of v in G_S' is X. Since $|X| = d$ and the in-edges of v are not contained in the same M_i in G_S' from Property 4(a), there are d disjoint walks from every node in X to v in G_S''. From this and $|F| = d-1$, v is reachable from some node $v' \in X$ in $G_S'' - F$. It is clear that $v' \in V' - Y$. Since it was proved that there is a walk from any remaining $u \in V'$ to any remaining $v' \in V' - Y$ in $G_S'' - F$, there is a walk $u, \ldots, v', \ldots, v$ in $G_S'' - F$.

Since there is an edge from w_i to w_j for all i, j, $i \neq j$, the proof of this theorem will be completed if it can be shown that there is a walk between any $u \in V' - F_V$ and any $w_i \in W - F_W$ in $G_S'' - F$. First, a walk from u to w_i will be shown. Let I be $\{j \mid w_j \in W - F_W\}$, from Property 4(c),

$$|P(W - F_W) \cap V'| = \left| \bigcup_{j \in I} X_j \right| \geqslant d - t + |I|.$$

From $|I| = |W - F_W| = |W| - |F_W| = t - k$ and $s + k = d - 1$, we get $|P(W - F_W) \cap V'| \geqslant d - t + |I| = d - k = s + 1$. From this and $|F_V| = s$, $|P(W - F_W) \cap (V' - F_V)| \geqslant |P(W - F_W) \cap V'| - |F_V| \geqslant 1$. Namely, there is an edge from some $v \in V' - F_V$ to some $w_l \in W - F_W$ in $G_S'' - F$. Since there is an edge from w_l to w_i, and it was proved that there is a walk between any two nodes in $V' - F_V$ in $G_S'' - F$, there is a walk $u, \ldots, v, w_l, w_i$ in $G_S'' - F$. The case from any $w_i \in W - F_W$ to any $u \in V' - F_V$ is similar. □

Acknowledgements

The authors wish to express their gratitude to Prof. J.-C. Bermond and Dr. H. Ichikawa for their helpful comments and suggestions.

References

[1] J.N. Ayoub and I.T. Frisch, Optimally invulnerable directed communication networks, IEEE Trans. Comm. 18 (1970) 484–489.

[2] J.-C. Bermond, N. Homobono and C. Peyrat, Large fault-tolerant interconnection networks, Graphs Combin. 5 (1989) 107–123.

[3] W.G. Bridge and S. Toueg, On the impossibility of directed Moore graphs, J. Combin. Theory Ser. B 29 (1980) 339–341.

[4] D.Z. Du, D.F. Hsu and G.W. Peck, Connectivity of consecutive-d digraphs, Discrete Appl. Math., submitted.
[5] D.Z. Du and F.K. Hwang, Generalized de Bruijn digraphs, Networks 18 (1988) 27–38.
[6] M.A. Fiol, J.L.A. Yebra and I. Alegre, Line digraph iterations and the (d, k) digraph problem, IEEE Trans. Comput. 33 (1984) 400–403.
[7] M. Imase and M. Itoh, Design to minimize a diameter on building block network, IEEE Trans. Comput. 30 (1981) 439–442.
[8] M. Imase and M. Itoh, A design for directed graphs with minimum diameter, IEEE Trans. Comput. 32 (1983) 782–784.
[9] M. Imase, T. Soneoka and K. Okada, Connectivity of regular directed graphs with small diameter, IEEE Trans. Comput. 34 (1985) 267–273.
[10] M. Mora, O. Serra and M.A. Fiol, General properties of c-circulant digraphs. Ars Combin. 25C (1988) 241–252.
[11] S.M. Reddy, D.K. Pradhan and J.G. Kuhl, Directed graphs with minimal diameter and maximal connectivity, Technical Report, School of Engineering, Oakland University, Rochester, MI (1980).
[12] U. Schumacher, An algorithm for construction of k-connected graph with minimum number of edges and quasiminimal diameter, Networks 14 (1984) 63–74.
[13] A. Sengupta, P.D. Joshi and S. Bandyopadhyay, A synthesis approach to design optimally fault tolerant network architecture, in: Proceedings of the 17th International Symposium on Fault-Tolerant Computing (1987) 198–203.
[14] T. Soneoka, H. Nakada and M. Imase, A design of d-connected digraph with minimum number of edges and quasiminimal diameter I, Discrete Appl. Math. 27 (1990) 255–265.